GCD AND LCM-LIKE IDENTITIES FOR IDEALS IN COMMUTATIVE RINGS

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Abstract. Let $A_1, \ldots, A_n$ ($n \geq 2$) be ideals of a commutative ring $R$. Let $G(k)$ (resp., $L(k)$) denote the product of all the sums (resp., intersections) of $k$ of the ideals. Then we have

$$L(n)G(2)G(4) \cdots G(2 \left\lceil \frac{n}{2} \right\rceil) \subseteq G(1)G(3) \cdots G(2 \left\lfloor \frac{n}{2} \right\rfloor - 1).$$

In the case $R$ is an arithmetical ring we have equality. In the case $R$ is a Prüfer ring, the equality holds if at least $n - 1$ of the ideals $A_1, \ldots, A_n$ are regular. In these two cases we also have

$$G(n)L(2)L(4) \cdots L(2 \left\lfloor \frac{n}{2} \right\rfloor) = L(1)L(3) \cdots L(2 \left\lfloor \frac{n}{2} \right\rfloor - 1).$$

Related equalities are given for Prüfer $v$-multiplication domains and formulas relating GCD’s and LCM’s in a GCD domain generalizing $\gcd(a_1, a_2)\lcm(a_1, a_2) = a_1a_2$ are given.

1. Introduction

It is well known that for natural numbers $a_1$ and $a_2$ (or more generally for nonzero elements of a PID)

$$\gcd(a_1, a_2)\lcm(a_1, a_2) = a_1a_2.$$ 

In terms of ideals this says that

$$((a_1) + (a_2))((a_1) \cap (a_2)) = (a_1)(a_2)$$ \n
or, since all ideals are principal, that

$$(A_1 + A_2)(A_1 \cap A_2) = A_1A_2$$ \n
for all ideals $A_1$, $A_2$ of a PID. In fact, this ideal equality nearly characterizes PIDs among integral domains. Indeed, an integral domain $R$ is a Prüfer domain (i.e., every nonzero finitely generated ideal is invertible) if and only if

$$(A_1 + A_2)(A_1 \cap A_2) = A_1A_2$$ \n
for all ideals $A_1$, $A_2$ of $R$ (see Theorem 2.6).
Less well known is the following formula for the gcd of $n$ ($n \geq 2$) natural numbers $a_1, \ldots, a_n$ in terms of their lcms:

$$(\text{GCD})_n \quad \gcd(a_1, \ldots, a_n) \prod_{2 \leq 2k \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k} \leq n} \operatorname{lcm}(a_{i_1}, \ldots, a_{i_{2k}})$$

$$= a_1 \cdots a_n \prod_{2 \leq 2k+1 \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k+1} \leq n} \operatorname{lcm}(a_{i_1}, \ldots, a_{i_{2k+1}})$$

and the formula obtained by interchanging gcd and lcm:

$$(\text{LCM})_n \quad \operatorname{lcm}(a_1, \ldots, a_n) \prod_{2 \leq 2k \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k} \leq n} \gcd(a_{i_1}, \ldots, a_{i_{2k}})$$

$$= a_1 \cdots a_n \prod_{2 \leq 2k+1 \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k+1} \leq n} \gcd(a_{i_1}, \ldots, a_{i_{2k+1}})$$

(see for example [9]). It is easy to prove that the simple “formula”

$$\gcd(a_1, \ldots, a_n) \operatorname{lcm}(a_1, \ldots, a_n) = a_1 \cdots a_n$$

holds precisely when each prime divides at most two of the $a_i$’s.

The corresponding ideal formulation is as follows. Let $R$ be a commutative ring and let $A_1, \ldots, A_n$ ($n \geq 2$) be ideals of $R$. For $1 \leq k \leq n$ put

$$G(k) := G(k; A_1, \ldots, A_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (A_{i_1} + \cdots + A_{i_k}),$$

$$L(k) := L(k; A_1, \ldots, A_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (A_{i_1} \cap \cdots \cap A_{i_k})$$

(so $G(1) = L(1) = A_1 \cdots A_n$, $G(n) = A_1 + \cdots + A_n$, $L(n) = A_1 \cap \cdots \cap A_n$).

**Definition 1.1.** The ring $R$ satisfies $(\ast)_n$ for ideals $A_1, \ldots, A_n$ of $R$ ($n \geq 2$) if

$$(\ast)_n \quad G(n) \prod_{2 \leq 2k \leq n} L(2k) = \prod_{1 \leq 2k+1 \leq n} L(2k+1)$$

and satisfies $(\ast\ast)_n$ for ideals $A_1, \ldots, A_n$ of $R$ ($n \geq 2$) if

$$(\ast\ast)_n \quad L(n) \prod_{2 \leq 2k \leq n} G(2k) = \prod_{1 \leq 2k+1 \leq n} G(2k+1).$$

Using the ceiling function and the floor function, we may express these as follows:

$$(\ast)_n \quad G(n)L(2)L(4) \cdots L(2 \lfloor n/2 \rfloor) = L(1)L(3) \cdots L(2 \lfloor n/2 \rfloor - 1),$$

$$(\ast\ast)_n \quad L(n)G(2)G(4) \cdots G(2 \lfloor n/2 \rfloor) = G(1)G(3) \cdots G(2 \lfloor n/2 \rfloor - 1).$$
Note that $(*)_2$ reduces to $(A_1 + A_2)(A_1 \cap A_2) = A_1A_2$ and $(**)_2$ reduces to $(A_1 \cap A_2)(A_1 + A_2) = A_1A_2$. We are taking $n \geq 2$ as the properties $(*)_1$ and $(**)_1$ are simply $A_1 = A_1$ which is always true.

A commutative ring $R$ is called a \textit{chained} ring (resp., \textit{arithmetical ring}) if the lattice of ideals of $R$ is a chain (resp. distributive). Here a \textit{chain} means a totally ordered set. So an integral domain is a chained ring if and only if it is a valuation domain. It is well known that $R$ is an arithmetical ring if and only if $R_M$ is a chained ring for each maximal ideal $M$ of $R$. An integral domain is a \textit{Prüfer domain} if every nonzero finitely generated ideal is invertible. $R$ is a Prüfer domain if and only if $R_M$ is a valuation domain for each maximal ideal $M$ of $R$. Finally, $R$ is a \textit{Prüfer ring} if every finitely generated regular ideal is invertible. Here, an element is \textit{regular} if it is not a zero-divisor and an ideal is \textit{regular} if it contains a regular element.

We show (Theorem 2.4) if $R$ is an arithmetical ring, then $(*)_n$ and $(**)_n$ hold for all ideals $A_1, \ldots, A_n$ of $R$ and that, if $R$ is a Prüfer ring, then $(*)_n$ and $(**)_n$ hold for all ideals $A_1, \ldots, A_n$ of $R$ when at least $n-1$ of them are regular.

We also prove that $(\text{GCD})_n$ and $(\text{LCM})_n$ hold for any GCD domain. Recall that an integral domain is a GCD domain if any two elements have a GCD, or equivalently, any two elements have an LCM. We prove this in the more general setting of a Prüfer \textit{v}-multiplication domain (PVMD). To define a PVMD we need a little back ground on the \textit{v}-operation and \textit{t}-operation.

Let $R$ be an integral domain with quotient field $K$. For a nonzero fractional ideal $I$ of $R$, $I_v := (I^{-1})^{-1}$ where $I^{-1} = \{ x \in K : xI \subseteq R \}$. It is well known (and easy to show) that $I_v = \bigcap \{ Rx : Rx \supseteq I, x \in K \}$. Then

$$I_t := \bigcup \{(a_1, \ldots, a_n)_v : a_1, \ldots, a_n \in I \setminus \{0\} \}.$$ 

So, for $I$ finitely generated, we have $I_t = I_v$. A nonzero fractional ideal $I$ is \textit{t-invertible} if $(II^{-1})_t = R$. Finally, $R$ is PVMD if every nonzero finitely generated ideal of $R$ is t-invertible. It is well known that an integral domain $R$ is a GCD domain if and only if $R$ is a PVMD in which every t-invertible ideal is principal. Thus, for nonzero $a_1, \ldots, a_n$ in a GCD domain, $(a_1, \ldots, a_n)_t$ is a principal ideal.

Suppose that $R$ is an integral domain. Then nonzero $a_1, \ldots, a_n \in R$ have an LCM if and only if $(a_1) \cap \cdots \cap (a_n)$ is principal and in this case $(a_1) \cap \cdots \cap (a_n) = (d)$ where $d = \text{lcm}(a_1, \ldots, a_n)$. Note that LCM’s and GCD’s are only defined up to unit multiple. If $(a_1, \ldots, a_n)_t$ is the
principal ideal $(d)$, then the GCD exists and $\gcd(a_1, \ldots, a_n) = d$. If $R$ is a GCD domain, then $\gcd(a_1, \ldots, a_n) = d$ exists and, since $(a_1, \ldots, a_n)_t$ is principal, we have $(d) = (a_1, \ldots, a_n)_t$.

We show (Theorem 2.4) that if $R$ is PVMD,

$$\left( G(n) \prod_{2 \leq 2k \leq n} L(2k) \right)_t = \left( \prod_{1 \leq 2k+1 \leq n} L(2k+1) \right)_t,$$

$$\left( L(n) \prod_{2 \leq 2k \leq n} G(2k) \right)_t = \left( \prod_{1 \leq 2k+1 \leq n} G(2k+1) \right)_t.$$

Then taking $R$ to be GCD domain and $A_1 = (a_1), \ldots, A_n = (a_n)$, we obtain $(\text{GCD})_n$ and $(\text{LCM})_n$ for $R$ (Theorem 2.8).

A good reference for multiplicative ideal theory including Prüfer domains, GCD domains, and $v$-operation is [5]. For results on Prüfer domains and Prüfer rings, see [8]. For a good survey of the $v$-operation and $t$-operation, $t$-invertibility, and PVMDs, see [10]. Finally, see [1] for a survey of GCD domains and related topics including PVMDs.

Thus neither $(\star)_n$ nor $(\star\star)_n$ always holds. In [2], however, we show that the one-sided inclusion

$$(\uparrow_n) \quad L(n) \prod_{2 \leq 2k \leq n} G(2k) \subseteq \prod_{1 \leq 2k+1 \leq n} G(2k+1)$$

holds for general commutative rings (which may not have an identity). Indeed, this holds not only for ideal lattices of rings but in the quite general setting of a commutative multiplication lattice.

Finally, in Section 3 we give some examples to illustrate results from Section 2.

2. Formulas $(\star)_n$ and $(\star\star)_n$ for ideals

In this section we prove various results concerning $(\star)_n$ and $(\star\star)_n$ for ideals of commutative rings that are mentioned in the Introduction.

We begin with the following fundamental lemma.

**Lemma 2.1.** Let $R$ be a commutative ring (not necessarily with identity) and let $A_1, \ldots, A_n$ ($n \geq 2$) be ideals of $R$. Suppose that $\{A_1, \ldots, A_n\}$ has a maximum (resp., minimum) element. Then $(\star)_n$ (resp., $(\star\star)_n$) holds for $A_1, \ldots, A_n$.

**Proof.** We prove the case for $(\star)_n$. The proof for $(\star\star)_n$ is similar. Without loss of generality we may assume that $A_1, \ldots, A_{n-1} \subseteq A_n$. Here $G(n) = A_1 + \cdots + A_n = A_n$, $L(1) = A_1 \cdots A_n$ and $L(n) = \ldots$
Let \( L(n; A_1, \ldots, A_n) = L(n-1; A_1, \ldots, A_{n-1}) \). For \( 2 \leq k \leq n-1 \),
\[
L(k) = L(k; A_1, \ldots, A_n) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (A_{i_1} \cap \cdots \cap A_{i_k})
\]
\[
= \prod_{1 \leq i_1 < \cdots < i_{k-1} \leq n-1} \prod_{i_k \leq i_{k+1} \leq n} (A_{i_1} \cap \cdots \cap A_{i_{k-1}})
\]
\[
= L(k; A_1, \ldots, A_{n-1})L(k-1; A_1, \ldots, A_{n-1}).
\]

For \( n \) even,
\[
G(n) \prod_{1 \leq k \leq n/2} L(2k)
\]
\[
= A_n \left( \prod_{1 \leq k \leq n/2-1} L(2k; A_1, \ldots, A_{n-1})L(2k-1; A_1, \ldots, A_{n-1}) \right)
\]
\[
\cdot L(n-1; A_1, \ldots, A_{n-1}) = A_n \prod_{1 \leq l \leq n-1} L(l; A_1, \ldots, A_{n-1}).
\]

On the other hand,
\[
\prod_{0 \leq k \leq n/2-1} L(2k+1) = L(1; A_1, \ldots, A_n)
\]
\[
\cdot \prod_{1 \leq k \leq n/2-1} L(2k+1; A_1, \ldots, A_{n-1})L(2k; A_1, \ldots, A_{n-1})
\]
\[
= A_n L(1; A_1, \ldots, A_{n-1}) \prod_{2 \leq l \leq n-1} L(l; A_1, \ldots, A_{n-1})
\]
\[
= A_n \prod_{1 \leq l \leq n-1} L(l; A_1, \ldots, A_{n-1}).
\]

Thus the two sides of (\(*\)) agree. The case where \( n \) is odd is similar. \( \square \)

**Lemma 2.2.** Let \( R \) be commutative ring with identity.

1. If \( A_1, \ldots, A_n \) are ideals \( R \), then \((*)_n \) (resp., \( **_n \)) holds for \( A_1, \ldots, A_n \) if and only if \((*)_n \) (resp., \( **_n \)) holds for the localizations \( A_{1M}, \ldots, A_{nM} \) for each maximal ideal \( M \) of \( R \).
2. The condition \((*)_n \) (resp., \( **_n \)) holds for all ideals of \( R \) if and only if \((*)_n \) (resp., \( **_n \)) holds for all ideals of \( R_M \) for each maximal ideal \( M \) of \( R \).
3. If \((*)_n \) (resp., \( **_n \)) holds for all ideals of \( R \) and \( S \) is a multiplicatively closed subset of \( R \), then \((*)_n \) (resp., \( **_n \)) holds for all ideals of \( R_S \).

**Proof.** This easily follows from the fact that localization preserves sums, products and intersections and that two ideals \( I \) and \( J \) of \( R \) are equal.
if and only if they are locally equal, that is, $I_M = J_M$ for each maximal ideal $M$ of $R$.

**Theorem 2.3.** Let $R$ be a commutative ring with identity and let $A_1, \ldots, A_n$ be ideals of $R$. Suppose that the set $\{A_{1M}, \ldots, A_{nM}\}$ of localizations forms a chain for each maximal ideal $M$ of $R$. Then we have the equalities $(*)_n$ and $(**)_n$ for $A_1, \ldots, A_n$.

**Proof.** This follows from Lemmas 2.1, 2.2.

The following is the main result of this section.

**Theorem 2.4.**

1. Suppose that $R$ is an arithmetical ring. Then $(*)_n$ and $(**)_n$ holds for all ideals $A_1, \ldots, A_n$ of $R$.

2. Suppose that $R$ is a Prüfer ring. Then $(*)_n$ and $(**)_n$ hold for all ideals $A_1, \ldots, A_n$ of $R$ where at least $n-1$ of them are regular.

3. Suppose that $R$ is PVMD and $A_1, \ldots, A_n$ are nonzero ideals of $R$. Then

   $$(G(n) \prod_{2 \leq 2k \leq n} L(2k))_t = \left( \prod_{1 \leq 2k+1 \leq n} L(2k+1) \right)_t,$$

   $$(L(n) \prod_{2 \leq 2k \leq n} G(2k))_t = \left( \prod_{1 \leq 2k+1 \leq n} G(2k+1) \right)_t.$$

   In particular, if $R$ is a Prüfer domain, $(*)_n$ and $(**)_n$ hold for all ideals $A_1, \ldots, A_n$ of $R$.

**Proof.**

1. Suppose that $R$ is arithmetical. Then for each maximal ideal $M$ of $R$, $R_M$ is a chained ring. So $\{A_{1M}, \ldots, A_{nM}\}$ is a chain. By Theorem 2.3, $(*)_n$ and $(**)_n$ hold for $A_1, \ldots, A_n$.

2. Suppose that $R$ is a Prüfer ring. Let $M$ be a maximal ideal of $R$. Let $R_{[M]}$ and $[M]R_{[M]}$ denote the large quotient ring of $R$ with respect to $M$ and the extension of $M$, respectively (see [8, p.234]). Since $R$ is a Prufer ring, $(R_{[M]}, [M]R_{[M]})$ is a valuation pair of the total quotient ring of $R$ [8, Theorem 10.18]. By [8, Exercise 10 (c), p. 248], $\{A_{1M}, \ldots, A_{nM}\}$ is a chain. Again by Theorem 2.3, $(*)_n$ and $(**)_n$ hold for $A_1, \ldots, A_n$.

3. For $f \in R[X]$, $A_f$ is the content of $f$, i.e. the ideal of $R$ generated by coefficients of $f$. Suppose that $R$ is PVMD. Put $R\{X\} = R[X]_{N_v}$ where $N_v$ is the multiplicatively closed subset \{f \in R[X] : (A_f)_v = R\} of $R[X]$. Since polynomial extensions and localization preserve sums, products and intersections, we
have

\[
\left( G(n) \prod_{2 \leq 2k \leq n} L(2k) \right) R\{X \}
\]

\[= \left( G(n; A_1, \ldots, A_n) \prod_{2 \leq 2k \leq n} L(2k; A_1, \ldots, A_n) \right) R\{X \}
\]

\[= G(n; A_1 R\{X \}, \ldots, A_n R\{X \}) \prod_{2 \leq 2k \leq n} L(2k; A_1 R\{X \}, \ldots, A_n R\{X \})
\]

and

\[
\left( \prod_{1 \leq 2k+1 \leq n} L(2k+1) \right) R\{X \}
\]

\[= \prod_{1 \leq 2k+1 \leq n} L(2k+1; A_1 R\{X \}, \ldots, A_n R\{X \}).
\]

Since \( R\{X \} \) is a Prüfer domain (even a Bézout domain) [7, Theorem 3.7], we have

\[
G(n; A_1 R\{X \}, \ldots, A_n R\{X \}) \prod_{2 \leq 2k \leq n} L(2k; A_1 R\{X \}, \ldots, A_n R\{X \})
\]

\[= \prod_{1 \leq 2k+1 \leq n} L(2k+1; A_1 R\{X \}, \ldots, A_n R\{X \}).
\]

But, for any nonzero ideal \( A \) of \( R \), we have \( AR\{X \} \cap R = A_t \) [7, Lemma 3.13]. Hence

\[
\left( G(n) \prod_{2 \leq 2k \leq n} L(2k) \right)_t = \left( G(n) \prod_{2 \leq 2k \leq n} L(2k) \right) R\{X \} \cap R
\]

\[= \left( \prod_{1 \leq 2k+1 \leq n} L(2k+1) \right) R\{X \} \cap R = \left( \prod_{1 \leq 2k+1 \leq n} L(2k+1) \right)_t.
\]

The corresponding \((**)_n\) identity is proved in a similar manner.

\[\square\]

**Theorem 2.5.** Let \( R \) be a commutative ring with identity. If \( R \) satisfies \((**)_n\) for all ideals of \( R \) for some \( n \geq 3 \), then \( R \) satisfies \((**)_2, \ldots, (**)_n-1\) for all ideals of \( R \).

**Proof.** Suppose that \( R \) satisfies \((**)_n\) for some \( n \geq 3 \):

\[
L(n) \prod_{2 \leq 2k \leq n} G(2k) = \prod_{1 \leq 2k+1 \leq n} G(2k+1)
\]

for all ideals \( A_1, \ldots, A_n \) of \( R \). Set \( A_n = R \) in the above equation. Then

\[
L(n; A_1, \ldots, A_{n-1}, R) = A_1 \cap \cdots \cap A_{n-1} \cap R = A_1 \cap \cdots \cap A_{n-1}
\]
\[ L(n; A_1, \ldots, A_{n-1}, R) = A_1 + \cdots + A_{n-1} + R = R. \]

For \( 1 \leq k < n \),
\[ G(k; A_1, \ldots, A_{k-1}, R) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} (A_{i_1} + \cdots + A_{i_k}) = G(k-1; A_1, \ldots, A_{n-1}). \]

So, substituting \( A_n = R \) in \((**)_n\), we obtain
\[ (**)_n \quad L(n-1) \prod_{2 \leq 2k \leq n-1} G(2k) = \prod_{1 \leq 2k+1 \leq n-1} G(2k+1). \]

Now \((**)_2, \ldots, (**)_n-1\) follow by reversed induction.

In Example 3.2, we show that \((*)_n \Rightarrow (**)_n\) for \( n \geq 3 \). Thus there is no analogous result for \((*)_n\). If in \((*)_n\) we set \( A_n = 0 \) (or any \( A_i = 0 \)), both sides collapse to 0 while if we set \( A_n = R \) (or any \( A_i = R \)), both sides become \( \prod_{1 \leq k \leq n-1} L(k; A_1, \ldots, A_{n-1}) \) (the analogous formula where \( A_i \) is deleted). Of course if in \((**)_n\) we set \( A_n = 0 \) (or any \( A_i = 0 \)), both sides collapse to 0.

**Theorem 2.6.** For a commutative ring \( R \) with identity, the following conditions are equivalent.

1. \( R \) is a Prüfer ring.
2. For each \( n \geq 2 \), \( R \) satisfies \((**)_n\) for all sets of \( n \) ideals \( A_1, \ldots, A_n \) where at least \( n-1 \) of them are regular.
3. For some \( n \geq 2 \), \( R \) satisfies \((**)_n\) for all sets of \( n \) regular ideals \( A_1, \ldots, A_n \).

**Proof.** (1) \( \Rightarrow \) (2): This is proved in Theorem 2.4.
(2) \( \Rightarrow \) (3): Clear.
(3) \( \Rightarrow \) (1): Suppose that \( R \) satisfies \((**)_n\) for all sets of \( n \) regular ideals \( A_1, \ldots, A_n \). Then by the proof of Theorem 2.5, \( R \) satisfies \((**)_2\) for all pairs of regular ideals \( A_1, A_2 \). Hence \( R \) is a Prüfer ring by [3, Theorem 4].

**Corollary 2.7.** Let \( R \) be a commutative ring with identity and let \( n \geq 2 \) be an integer. Suppose that \( R \) satisfies \((**)_n\) for each set \( A_1, \ldots, A_n \) of regular ideals of \( R \). Then \( R \) satisfies \((*)_m\) and \((**)_m\) for each integer \( m \geq 1 \) and ideals \( A_1, \ldots, A_m \) with at least \( m-1 \) of them regular. Hence if an integral domain satisfies \((**)_n\) for some integer \( n \geq 2 \), it satisfies \((*)_m\) and \((**)_m\) for all integers \( m \geq 1 \).
Proof. Suppose that \( R \) satisfies \((**)_n\) for some particular \( n \geq 2\) and all regular ideals \( A_1, \ldots, A_n \). By Theorem 2.6, \( R \) is a Prüfer ring. Then by Theorem 2.4, for each natural numbers \( m \) (the case \( m = 1 \) is trivial), \( R \) satisfies \((*)_m\) and \((**)_m\) for all sets of \( m \) ideals \( A_1, \ldots, A_m \) with at least \( m - 1 \) of them regular. \( \square \)

It goes back at least to Krull that an integral domain \( R \) is a Prüfer domain if and only if \( (A + B)(A \cap B) = AB \) for all ideals \( A \) and \( B \) of \( R \). Indeed, if we take \( A = (a) \) and \( B = (b) \) to be nonzero principal ideals, we have \( (a, b)(((a) \cap (b)) = (a)(b) \); so \( (a, b) \) is invertible, being a factor of a principal ideal. By induction, each finitely generated nonzero ideal is invertible; i.e. \( R \) is a Prüfer domain. The result that \( R \) is a Prüfer ring if and only if \( (A + B)(A \cap B) = AB \) for all ideals \( A \) and \( B \) with at least one regular appears to first be given in [4]. (Note that [4] uses the term “Prüfer ring” for what we have called an arithmetical ring.)

We next prove the formulas \((\text{GCD})_n\) and \((\text{LCM})_n\) for a GCD domain.

\[ \text{(GCD)}_n \quad \gcd(a_1, \ldots, a_n) \prod_{2 \leq 2k \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k} \leq n} \text{lcm}(a_{i_1}, \ldots, a_{i_{2k}}) \]
\[ = a_1 \cdots a_n \prod_{2 \leq 2k + 1 \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k+1} \leq n} \text{lcm}(a_{i_1}, \ldots, a_{i_{2k+1}}) \]
and
\[ \text{(LCM)}_n \quad \text{lcm}(a_1, \ldots, a_n) \prod_{2 \leq 2k \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k} \leq n} \gcd(a_{i_1}, \ldots, a_{i_{2k}}) \]
\[ = a_1 \cdots a_n \prod_{2 \leq 2k + 1 \leq n} \prod_{1 \leq i_1 < \cdots < i_{2k+1} \leq n} \gcd(a_{i_1}, \ldots, a_{i_{2k+1}}) \).

\[ \text{Proof.} \text{ Recall that a GCD domain is a PVMD with } (b_1) \cap \cdots \cap (b_n) \text{ and } (b_1, \ldots, b_n) \text{1 principal for all nonzero } b_1, \ldots, b_n \in R. \text{ Moreover, if } (b_1) \cap \cdots \cap (b_n) = (d) \text{ (resp., } (b_1 \cdots b_n) \ell = (e) \text{), then } d = \text{lcm}(b_1 \cdots b_n) \text{ (resp., } e = \text{gcd}(b_1 \cdots b_n)). \text{ Recall that lcm and gcd are only determined up to a unit multiple. Throughout the proof below, we will use the fact that } (AB)_t = (AB_t) = (A_t B_t), \text{ for nonzero ideals } A \text{ and } B \text{ of } R. \text{ Put } A_t = (a_t). \text{ Note that } \]
\[ G(n)_t = ((a_1) + \cdots + (a_n))_t = (\gcd(a_1, \ldots, a_n))_t, \]
\[ L(n)_t = ((a_1) \cap \cdots \cap (a_n))_t = (a_1) \cap \cdots \cap (a_n) \]
\[ = (\text{lcm}(a_1, \ldots, a_n))_t = L(n). \]
Also, \( G(1) = (a_1) \cdots (a_n) = (a_1 \cdots a_n) = L(1) \). For \( 1 < k < n \), we have
\[
G(k)_t = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} (a_{i_1}, \ldots, a_{i_k}) \right)_t = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} (a_{i_1}, \ldots, a_{i_k})_t \right)
\]
\[
= \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} (\gcd(a_{i_1}, \ldots, a_{i_k})) \right)_t = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} (\gcd(a_{i_1}, \ldots, a_{i_k})) \right)
\]
\[
= \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} \gcd(a_{i_1}, \ldots, a_{i_k}) \right)
\]
and
\[
L(k) = \prod_{1 \leq i_1 < \cdots < i_k \leq n} ((a_{i_1}) \cap \cdots \cap (a_{i_k}))
\]
\[
= \prod_{1 \leq i_1 < \cdots < i_k \leq n} (\text{lcm}(a_{i_1}, \ldots, a_{i_k})) = \left( \prod_{1 \leq i_1 < \cdots < i_k \leq n} \text{lcm}(a_{i_1}, \ldots, a_{i_k}) \right).
\]
(Since each \((a_{i_1}) \cap \cdots \cap (a_{i_k})\) and hence \(L(k)\) is principal, we have \(L(k)_t = L(k)\).) Now, since \(R\) is a PVMD, so we have
\[
\left( G(n) \prod L(2k) \right)_t = \left( \prod L(2k + 1) \right)_t
\]
and hence
\[
\left( G(n) \prod L(2k) \right)_t = \left( \prod L(2k + 1) \right)_t.
\]
Since both sides of the equality \(G(n) \prod L(2k) = \prod L(2k + 1)\) as ideals. This proves the first equality (up to unit multiplication). The proof of the second is similar. \(\Box\)

**Corollary 2.9.** Let \(R\) be an integral domain with the property that \(R_M\) is a GCD-domain for each maximal ideal \(M\) of \(R\). Then for each \(n \geq 2\),
\[
G(n) \prod_{2 \leq 2k \leq n} L(2k) \subseteq \prod_{2 \leq 2k + 1 \leq n} L(2k + 1)
\]
holds for all locally principal (e.g. invertible) ideals \(A_1, \ldots, A_n\) of \(R\).

**Proof.** Since it suffices to prove the containment locally (Lemma 2.2), we can assume that \(R\) is a quasi-local GCD domain and that the ideals \(A_1, \ldots, A_n\) are principal. If one of the \(A_i = 0\), both sides of the containment reduce to 0. So we can assume that \(A_1, \ldots, A_n\) are nonzero. Now from the proof of Theorem 2.8, we have \(G(n) \prod L(2k) = \prod L(2k + 1)\). Since \(G(n) \subseteq G(n)_t\) the result follows. \(\Box\)

Example 3.3 shows that the containment in Corollary 2.9 may be strict.
3. Examples

In this section we give some examples which illustrate the results of Section 2.

Example 3.1 \((*)_n \iff (**)_n \text{ for } n \geq 3\). Let \((R, M)\) be a quasi-local ring with \(M^n = 0 \ (n \geq 2)\). We claim that \(R\) satisfies \((*)_n\), but need not satisfy \((**)_n\).

Let \(A_1, \ldots, A_n\) be ideals of \(R\). If each \(A_i\) is a proper ideal, then \(R\) satisfies \((*)_n\) and \((**)_n\) for \(A_1, \ldots, A_n\) as both sides reduce to 0. Suppose some \(A_i = R\), then \((*)_n\) holds as both sides become

\[
\prod_{1 \leq k \leq n-1} L(k; A_1, \ldots, \hat{A}_i, \ldots, A_n)
\]

(see the paragraph after the proof of Theorem 2.5). So, \((*)_n\) holds for \(R\). By similar reasoning \(R\) satisfies \((**)_m\) for \(m \geq n\).

Now for \(n = 2\), \((*)_2\) holds and hence so does \((**)_2\). Suppose \(n \geq 3\) and consider the special case \(R_n = k[X, Y]/(X, Y)^n\) for a field \(k\). So \(R_n\) satisfies \((*)_n\) for each \(n \geq 2\). However, \(R_n\) does not satisfy \((**)_m\) for any \(m \geq 2\). For, if \(R_n\) satisfies \((**)_m\) for some \(m \geq 2\), then \(R_n\) satisfies \((**)_2\) by Theorem 2.5. This contradicts the strict inclusion

\[
(X, Y)((X) \cap (Y)) = (X, Y)(X Y) \subsetneq (X)(Y)
\]

Note that for \(n = 3\), \(R_n\) satisfies \((*)_3\) but not \((*)_2\). This is a counterexample of the \((*)\) analogue of Theorem 2.5. We generalize this in the next example.

Example 3.2 \((*)_n \iff (**)_n-1 \text{ for } n \geq 3\). While \((**)_n\) implies \((**)_n-1\) for \(n \geq 3\), we show that this need not be the case for \((*)_n\). Take \(R_n = k[X_1, \ldots, X_n]/(X_1, \ldots, X_n)^n\) for a field \(k\) \((n \geq 3)\). So \(R_n\) satisfies \((*)_n\) by Example 3.1. In fact \(R_n\) satisfies \((*)_m\) for each \(m \geq n\). However, for \(2 \leq m < n\), \((*)_m\) fails for \(A_1 = (X_1), \ldots, A_m = (X_m)\). The proof is essentially the same as the proof that

\[
G(n) \prod_{2 \leq 2k \leq n} L(2k) \subsetneq \prod_{1 \leq 2k+1 \leq n} L(2k+1)
\]

in \(k[X_1, X_2, \ldots]\) given in Example 3.3. So \(R_n\) satisfies \((*)_m\) precisely for \(m \geq n\).

Example 3.3 (A ring with \(G(n) \prod_{2 \leq 2k \leq n} L(2k) \subsetneq \prod_{1 \leq 2k+1 \leq n} L(2k+1)\) for all \(n \geq 2\). Put \(R = k[X_1, X_2, \ldots]\) for a field \(k\) and take \(A_i = (X_i)\). Note that \((X_{i_1}) \cap \cdots \cap (X_{i_k}) = (X_{i_1} \cdots X_{i_k})\). Then \(G(n) = (X_1, \ldots, X_n)\)
and \( L(i) = (X_1 \cdots X_n)^{\binom{n-1}{i}} \). It follows that
\[
\prod_{2 \leq 2k \leq n} L(2k) = \prod_{2 \leq 2k \leq n} (X_1 \cdots X_n)^{\binom{n-1}{2k-1}} = (X_1 \cdots X_n)^{\sum_{2 \leq 2k \leq n} \binom{n-1}{2k-1}},
\]
\[
\prod_{1 \leq 2k+1 \leq n} L(2k+1) = \prod_{1 \leq 2k+1 \leq n} (X_1 \cdots X_n)^{\binom{n-1}{2k}} = (X_1 \cdots X_n)^{\sum_{1 \leq 2k+1 \leq n} \binom{n-1}{2k}}.
\]
Since \( \sum_{i=0}^{n-1} \binom{n-1}{i} = 0 \), we have \( \prod_{2 \leq 2k \leq n} L(2k) = \prod_{1 \leq 2k+1 \leq n} L(2k+1) \) and hence \( G(n) \prod_{2 \leq 2k \leq n} L(2k) \subseteq \prod_{1 \leq 2k+1 \leq n} L(2k+1) \).

Note that \( R \) is a UFD and hence a GCD domain. So Corollary 2.9 gives that
\[
G(n) \prod_{2 \leq 2k \leq n} (2k) \subseteq \prod_{1 \leq 2k+1 \leq n} (2k+1).
\]
This example shows that containment may be strict.

We previously mentioned that in \([2]\) we show that for any commutative ring we have \( L(n) \prod_{2 \leq 2k \leq n} G(2k) \subseteq \prod_{1 \leq 2k+1 \leq n} G(2k+1) \). In \([2]\) we give examples to show that this inclusion may be strict for all \( n \geq 2 \) and that in general any relation between \( G(3)L(2) \) and \( L(1)L(3) \) is possible.

References

GCD AND LCM-LIKE IDENTITIES FOR IDEALS

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